Exercise 21

(a) Taking $c = L = 1$, plot several snapshots of the vibrating string in Exercise 15. Observe that the point at $x = \frac{1}{2}$ never moves. Prove this last observation.

(b) Find initial data for which the point $x = \frac{1}{3}$ never moves. Show that the point $x = \frac{2}{3}$ is also fixed.

Solution

The initial boundary value problem from Exercise 15 is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \ -\infty < t < \infty$$

$$u(x,0) = \sin \frac{2\pi x}{L}$$

$$\frac{\partial u}{\partial t}(x,0) = 0$$

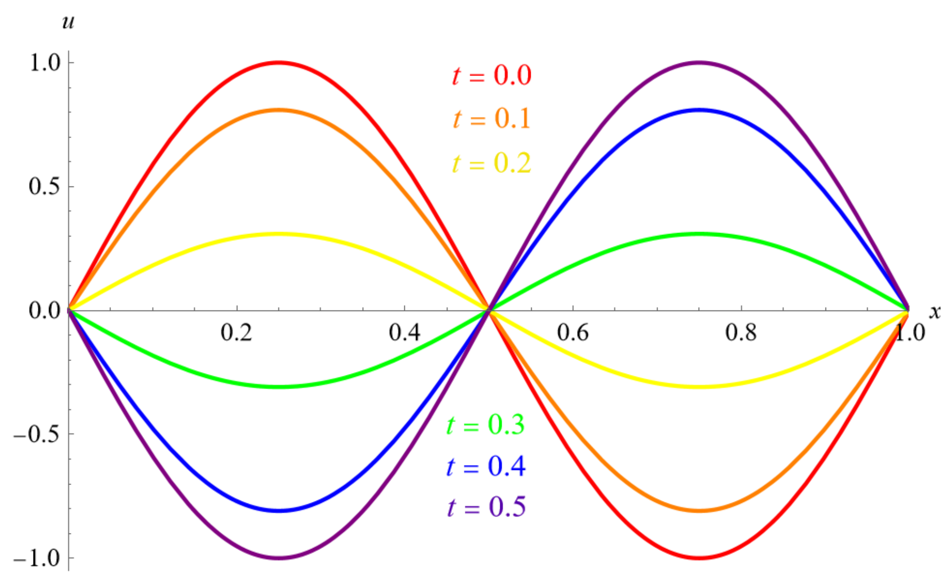
$$u(0,t) = 0$$

$$u(L,t) = 0,$$

and its solution is

$$u(x,t) = \sin \frac{2\pi x}{L} \cos \frac{2\pi c t}{L}.$$

Below is a plot of u versus x over $0 < x < 1$ at several times with $c = 1$ and $L = 1$.



Notice that u is zero at $x = \frac{1}{2}$ at all times. Plug in $c = 1$ and $L = 1$ and $x = \frac{1}{2}$ in the solution.

$$u\left(\frac{1}{2}, t\right) = \sin \pi \cos 2\pi t = 0$$

This is zero regardless of what t is.

The initial value problem for which the solution doesn't move at $x = \frac{1}{3}$ and $x = \frac{2}{3}$ is

$$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2}, \quad 0 < x < L, \ -\infty < t < \infty$$

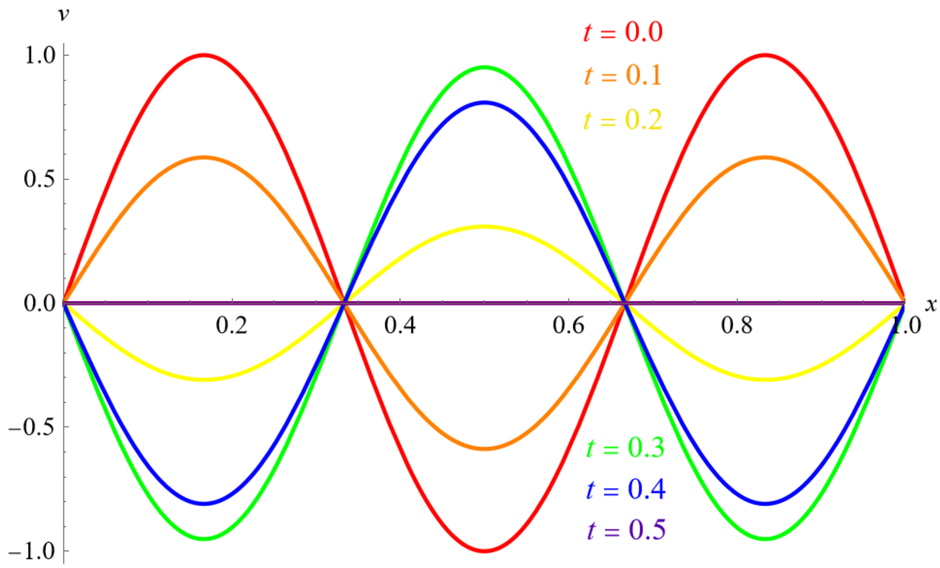
$$v(x, 0) = \sin \frac{3\pi x}{L}$$

$$\frac{\partial v}{\partial t}(x, 0) = 0.$$

Its solution is

$$v(x, t) = \sin \frac{3\pi x}{L} \cos \frac{3\pi c t}{L}.$$

Below is a plot of v versus x over $0 < x < 1$ at several times with $c = 1$ and $L = 1$.



Notice that v is zero at $x = \frac{1}{3}$ and $x = \frac{2}{3}$ at all times. Plug in $c = 1$ and $L = 1$ and $x = \frac{1}{3}$ in the solution.

$$v\left(\frac{1}{3}, t\right) = \sin \pi \cos 3\pi t = 0$$

Plug in $c = 1$ and $L = 1$ and $x = \frac{2}{3}$ in the solution.

$$v\left(\frac{2}{3}, t\right) = \sin 2\pi \cos 3\pi t = 0$$

These are zero regardless of what t is.